Geometric Bliss Beyond the Circle Kiss

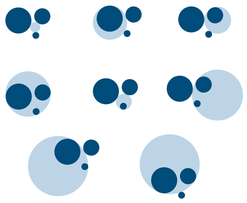
Maja Andrić, Dominka Jukić and Neda Lovričević

The Apollonius Problem and Geometric Inversion

Apollonius of Perga (262–190 BC), renowned as the Great Geometer, considered in his work *De Tactionibus* the problem of constructing, by Euclidean means, a circle tangent simultaneously to three given geometric objects.
These objects may be a point (P), a line (L), or a circle (C). Using the formula for the number of combinations with repetition of the third class of a multiset with three distinct elements, it follows that the total number of possible tangency configurations is $\binom{3+3-1}{3} = 10$.

Accordingly, the sought circle may be tangent to one of the following combinations:

Configuration	Description
P-P-P	3 points
P-P-L	2 points and 1 line
P-P-C	2 points and 1 circle
P-L-L	1 point and 2 lines
P-L-C	1 point, 1 line and 1 circle
P-C-C	1 point and 2 circles
L-L-L	3 lines
L-L-C	2 lines and 1 circle
L-C-C	1 line and 2 circles
C-C-C	3 circles



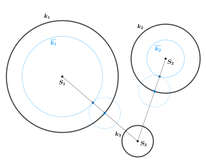
The final configuration (C–C–C) is commonly known as Apollonius' problem:

"How can a circle tangent to three given circles be constructed?"

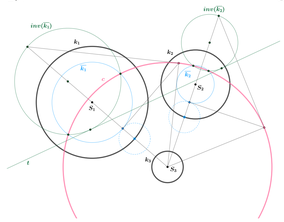
Since the sought circle can touch each given circle either externally or internally, there are $2^3 = 8$ possible solutions. These eight cases are illustrated for three pairwise disjoint circles.

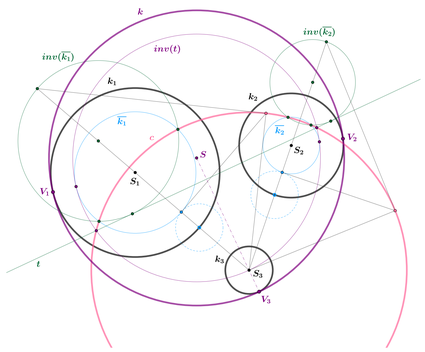
Apollonius' original straightedge-and-compass construction has not survived. The most elegant solution to the C–C–C case is achieved using geometric inversion, introduced in the 19th century. The fundamental idea is to map one of the given circles into a line, transforming the problem into a simpler sequence of elementary geometric constructions.
The construction procedure based on this approach is illustrated below.

Offset the circles inward by the radius of the smallest circle: $k_1 \longrightarrow \overline{k_1}, \quad k_2 \longrightarrow \overline{k_2}$.



Let c be the inversion circle centred at S_3 with arbitrary radius. Determine the images $inv(\overline{k_1})$ and $inv(\overline{k_2})$, and construct their common tangent t.





Determine the image $inv(t)$ of the line t under the inversion.

Offset the resulting circle outward by the radius of the smallest circle:

$$inv(t) \longrightarrow k,$$

obtaining the required solution.

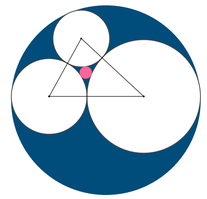
Frederick Soddy: The Kiss Precise

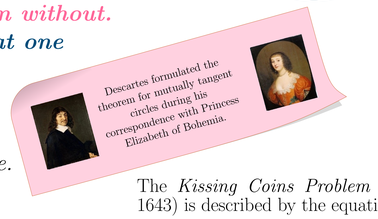
(Nature, 137, p. 1021, 1936)



Frederick Soddy (1877–1956) was an English chemist who was awarded the 1921 Nobel Prize in Chemistry "for his contributions to our knowledge of the chemistry of radioactive substances, and his investigations into the origin and nature of isotopes." In 1936, his poem *The Kiss Precise* was published in the scientific journal *Nature*, offering a poetic formulation of Descartes' theorem. The poem concerns a special case of Apollonius' problem involving three mutually tangent circles and a circle tangent to all three. Depending on the tangency type, two solutions may arise. Following Soddy's poetic treatment of the subject, the two possible solutions became known in the mathematical literature as *Soddy's circles*.

For pairs of lips to kiss maybe
Involves no trigonometry.
This not so when four circles kiss
Each one the other three.
To bring this off the four must be
As three in one or one in three.
If one in three, beyond a doubt
Each gets three kisses from without.
If three in one, then is that one
Thrice kissed internally.





Four circles to the kissing come.
The smaller are the benter.
The bend is just the inverse of
The distance from the center.
Though their intrigue left Euclid dumb
There's now no need for rule of thumb.
Since zero bend's a dead straight line
And concave bends have minus sign,
The sum of the squares of all four bends
Is half the square of their sum.

The *Kissing Coins Problem* (R. Descartes, 1643) is described by the equation

$$(k_1 + k_2 + k_3 + k_4)^2 = 2\left(k_1^2 + k_2^2 + k_3^2 + k_4^2\right),$$

where $k_i = \frac{1}{r_i}$ denote the curvatures of the circles with corresponding radii r_i. It follows that

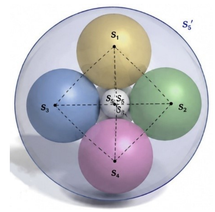
$$k_4 = k_1 + k_2 + k_3 \pm 2\sqrt{k_1k_2 + k_2k_3 + k_3k_1},$$

where the two solutions correspond to the curvatures of the smaller and the larger circle tangent to the given circles.

To spy out spherical affairs
An oscular surveyor
Might find the task laborious,
The sphere is much the gayer,
And now besides the pair of pairs
A fifth sphere in the kissing shares.
Yet, signs and zero as before,
For each to kiss the other four
The square of the sum of all five bends
Is thrice the sum of their squares.

Soddy extended the planar problem to the three-dimensional Euclidean case of five mutually tangent spheres:

$$(k_1 + k_2 + k_3 + k_4 + k_5)^2 = 3\left(k_1^2 + k_2^2 + k_3^2 + k_4^2 + k_5^2\right).$$



The Kiss Precise (Generalized)

(T. Gosset, Nature, 139, p. 62, 1937)

And let us not confine our cares
To simple circles, planes and spheres,
But rise to hyper flats and bends
Where kissing multiple appears,
In n-ic space the kissing pairs
Are hyperspheres, and Truth declares –
As $n+2$ such osculate
Each with an $n+1$ fold mate
The square of the sum of all the bends
Is n times the sum of their squares.

Thorold Gosset (1869–1962), an English lawyer and mathematician, published the fourth stanza as a continuation of Soddy's *The Kiss Precise*. He generalized Descartes' theorem to $(n-1)$-spheres in an n-dimensional Euclidean space. For $n+2$ mutually tangent hyperspheres,

$$\left(\sum_{i=1}^{n+2} k_i\right)^2 = n\sum_{i=1}^{n+2} k_i^2.$$

. . . and how might Soddy have written it?

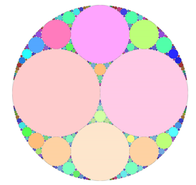
Beyond the bounds of three we climb,
The law survives the test of time;
Where higher worlds unfold to view,
The selfsame truth appears anew.
Let $(n+2)$ each kiss the rest,
In (n)-fold space they pass the test;
The ancient truth still rises higher
As simple circles first revealed:
The squared sum stands, unfailingly,
(n) times the squares, eternally.

Our aim was therefore to reconstruct a fourth stanza that faithfully imitates Soddy's original metre, rhyme scheme, and poetic diction, as though it had formed part of his 1936 poem.

The result is the following poetic formulation of the generalization—an imaginative reconstruction produced through human–AI collaboration, intended as a tribute to Frederick Soddy.

Apollonian Gasket

Three mutually tangent circles determine two additional circles tangent to all three, one externally and the other internally. Repeating this construction by inserting circles into every curvilinear triangular gap generates an infinite fractal known as the *Apollonian gasket*. It is a classical example of circle packing and extends naturally to higher-dimensional sphere packings.



Geometric Bliss: Still Unfolding

The Apollonius problem remains an active area of research. For example, Lagarias, Mallows and Wilks (*Beyond the Descartes Circle Theorem*, 2002) presented a unified treatment of the Apollonius problem in spaces of constant curvature κ, where $\kappa = 0$, $\kappa > 0$, and $\kappa < 0$ correspond to Euclidean, spherical, and hyperbolic geometries, respectively. More recently, Mathews and Zymaris (*Spinors and the Descartes Circle Theorem*, 2025) employed spinors to describe *n-flower* configurations of mutually tangent circles in hyperbolic geometry arising from the Apollonius problem.

[1] D. Palman, *Geometrijske konstrukcije*, Element,1996
[2] D. Gisch, J.M. Ribando, *Apollonius' Problem: A study of Solutions and Their Connections*, Am. J. Undergrad. Res. 3(1), 2024